Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(02)00108-9

J. Appl. Maths Mechs, Vol. 66, No. 5, pp. 767–773, 2002

0021-8928/02/$—see front matter

EXACT SOLUTIONS OF PROBLEMS OF THE THREE-DIMENSIONAL FLOW OF IDEAL VISCOUS INCOMPRESSIBLE FLUIDS NEAR CYLINDRICAL SURFACES†

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(*Received* 10 *August* 2000)

Three-dimensional flows of an incompressible fluid, the parameters of which depend on two coordinates and time, are considered. The stream surfaces of such flows are cylindrical. The equations of continuity and the Navier–Stokes equations can be transformed to relations, one of which is the equation for the stream function the other is the integral of the equations relating the pressure and the stream function, and the third is a linear equation for the projection of the velocity vector onto the axis parallel to the generatrix of the cylindrical surfaces. The problems of modelling the flows are considered on the basis of the exact solutions of the Navier–Stokes equations and Euler's equations using examples. Relations for the distribution of the flow parameters in the channel created by hyperbolical cylinders are derived for the case of unsteady inviscid flow. The streamlines of these flows are situated on the side surfaces of the hyperbolical cylinders and intercept the generatrices of the cylinders at certain indirect angles. The flow around a circular cylinder and the flow of fluid inside an elliptic cylinder are considered in the case of steady inviscid flow. The streamlines on the circular cylinder are arranged transverse to the cylinder (the projection of the velocity vector onto the coordinate axis, parallel to the generatrix of the cylinder, is equal to zero). Far from the cylinder the streamlines are also situated on a cylindrical surfaces, but not transverse to the cylinder, making certain indirect angles with the generatrix. Viscous three-dimensional flows, possessing a certain symmetry, are considered. In the case of radial symmetry the streamlines are helical lines. The non-planar Couette flow between parallel moving planes is characterized by the fact that the velocity vectors, being situated in the same plane, are collinear, while the velocity vectors in parallel planes are not collinear. Relations for viscous steady three-dimensional flows, using well-known relations, obtained for the stream function of two-dimensional flows, are given.

1. GENERAL RELATIONS

Consider the three-dimensional flow of a viscous incompressible fluid, the velocity vector of which $\mathbf{u} = (\mathbf{u}_1, u_2, u_3)$ depends on time t and two space coordinates: $u_k = u_k(t, x_1, x_2)$ $(k = 1, 2, 3)$ in a rectangular Cartesian system of coordinates x_1, x_2, x_3. Introducing the stream function $\psi = \psi(t, x_1, x_2)$ $(u_1 = \partial\psi/\partial x_2$, $u_2 = -\partial\psi/\partial x_1)$, from the Navier–Stokes equations, we have

$$\frac{\partial}{\partial x_l}\left(\frac{p}{\rho}+\frac{w^2}{2}\right) = (-1)^l \frac{\partial^2\psi}{\partial t\partial x_{3-l}} + \frac{\partial\psi}{\partial x_l}\Delta\psi + (-1)^{l+1}\nu\frac{\partial}{\partial x_{3-l}}\Delta\psi \equiv f_l(t, x_1, x_2), \quad l = 1, 2 \tag{1.1}$$

$$\frac{\partial}{\partial x_3}\frac{p}{\rho} = \frac{\partial u_3}{\partial t} + \frac{\partial\psi}{\partial x_2}\frac{\partial u_3}{\partial x_1} - \frac{\partial\psi}{\partial x_1}\frac{\partial u_3}{\partial x_2} - \nu\Delta u_3 \equiv f_3(t, x_1, x_2) \tag{1.2}$$

$$\nu = \text{const}, \quad \rho = \text{const}, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \quad w^2 = u_1^2 + u_2^2$$

Equations (1.1) give

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1} \tag{1.3}$$

which leads to the equation for the stream function

$$\frac{\partial\Delta\psi}{\partial t} + \frac{\partial\psi}{\partial x_2}\frac{\partial\Delta\psi}{\partial x_1} - \frac{\partial\psi}{\partial x_1}\frac{\partial\Delta\psi}{\partial x_2} = \nu\Delta\Delta\psi \tag{1.4}$$

†*Prikl. Mat. Mekh.* Vol. 66, No. 5, pp. 803–810, 2002.

Let $\psi = \psi(t, x_1, x_2)$ be the solution of Eq. (1.4). Then, considering Eqs (1.1) as a system of partial differential equations in the unknown function $p/\rho + w^2/2$, we obtain, after integrating Eq. (1.1) with $l = 1$

$$\frac{p}{\rho} + \frac{w^2}{2} = \int_{x_{10}}^{x_1} f_1(t, x_1, x_2)dx_1 + \gamma(t, x_1, x_2) \tag{1.5}$$

Taking into account Eq. (1.3), from Eq. (1.1) with $l = 2$ we obtain

$$\gamma = \int_{x_{20}}^{x_2} f_2(t, x_{10}, x_2)dx_2 + \beta(t, x_3) \tag{1.6}$$

In relations (1.5) and (1.6) x_{10} and x_{20} are certain known coordinates.

Expressing p/ρ from (1.5) and (1.6) and substituting the expression obtained into Eq. (1.2), we will have

$$\frac{\partial\beta(t, x_3)}{\partial x_3} = f_3(t, x_1, x_2)$$

This relation, the left-hand side of which depends only on t and x_3 and the right-hand side only on t and x_1, x_2, is only possible if both of its sides are independent of the coordinates and represent the same quantity, which depends only on t. We will denote this quantity by $c_1(t)$. This means that we must assume

$$\frac{\partial}{\partial x_3}\frac{p}{\rho} = c_1(t) \tag{1.7}$$

Now instead of Eqs (1.2) and (1.5) we will write

$$\frac{\partial u_3}{\partial t} + \frac{\partial\psi}{\partial x_2}\frac{\partial u_3}{\partial x_1} - \frac{\partial\psi}{\partial x_1}\frac{\partial u_3}{\partial x_2} = -c_1(t) + \nu\Delta u_3 \tag{1.8}$$

$$\frac{p}{\rho} + \frac{w^2}{2} = \int_{x_{10}}^{x_1}\left(\frac{\partial\psi}{\partial x_1}\Delta\psi + \nu\frac{\partial}{\partial x_2}\Delta\psi - \frac{\partial^2\psi}{\partial t\partial x_2}\right)dx_1 +$$
$$+\int_{x_{20}}^{x_2}\left(\frac{\partial\psi}{\partial x_2}\Delta\psi - \nu\frac{\partial}{\partial x_1}\Delta\psi + \frac{\partial^2\psi}{\partial t\partial x_1}\right)_{x_{10}}dx_2 + c_1(t)x_3 + c_2(t) \tag{1.9}$$

After integrating the left equality in the differential equations of the streamlines (t is a parameter)

$$\frac{\partial x_1}{\partial\psi/\partial x_2} = \frac{dx_2}{-\partial\psi/\partial x_1} = \frac{dx_3}{u_3(t, x_1, x_2)} \tag{1.10}$$

we obtain the first integral

$$\psi(t, x_1, x_2) = k_1 \tag{1.11}$$

This relation determines the implicitly given function x_2.

Now, taking the first and the last fraction in (1.10) we obtain

$$x_3 - \int \frac{u_3(t, x_1, x_2(t, x_1, k_1))}{\partial\psi/\partial x_2}dx_1 = k_2 \tag{1.12}$$

$$\left(\frac{\partial\psi}{\partial x_2} = \frac{\partial\psi}{\partial x_2}(t, x_1, x_2(t, x_1, k_1))\right)$$

The first integrals (1.11) and (1.12) form a general integral of system (1.10).

Equation (1.11) determines the cylindrical stream surface with generatrices parallel to the x_3 axis, at each instant of time. The position of the x_3 coordinate of the streamline for a given value of k_1 (i.e. for the given cylindrical surface) is determined from Eq. (1.12) at each instant of time. In this case we find the constant k_2 by assigning any section x_3 = const and a point on the stream surface (1.11) in this.

Hence, Eqs (1.4), (1.8) and (1.9) enable us to simulate the fluid flow in the vicinity of cylindrical surfaces.

In particular, let ψ be a function with separated variables.

$$\psi = A(t)\xi(x_1, x_2) \tag{1.13}$$

Then instead of Eq. (1.11) we will write

$$\xi(x_1, x_2) = k_1$$

In this case, the stream surfaces, defined by this equation for various k_1, do not vary with time.

The first integral in the steady-flow case will be

$$\psi(x_1, x_2) = k_1 \tag{1.14}$$

instead of (1.11).

2. THE SIMULATION OF UNSTEADY INVISCID FLOWS

In order to consider inviscid flow, terms with the multiplier ν must be dropped from Eqs (1.4) and (1.8).

The harmonic function ψ of the variables x_1 and x_2 is an obvious solution of Eq. (1.4) in this case

$$\Delta\psi = 0 \tag{2.1}$$

We take, as a simple example, the function

$$\psi = A(t)x_1x_2 \tag{2.2}$$

which satisfies Eq. (2.1). Integrating the following system of differential equations in the symmetric form

$$\frac{dt}{1} = \frac{dx_1}{A(t)x_1} = \frac{dx_2}{-A(t)x_2} = \frac{du_3}{-c_1(t)}$$

for the condition $A(t) = c_1(t)$, corresponding to the partial differential equation (1.8), we obtain the general solution of Eq. (1.8) in the given case

$$\Phi(\xi, \eta, \varsigma) = 0 \tag{2.3}$$

where

$$\xi = x_1x_2, \quad \eta = u_3 + \int c_1(t)dt, \quad \varsigma = u_3 + \ln|x_1|$$

In particular, if Φ is the linear function of the arguments η and ς we have

$$u_3 = f(x_1x_2) - \frac{1}{2}\left(\int c_1(t)dt + \ln|x_1|\right) \tag{2.4}$$

where f is an arbitrary continuously differentiable function.

Using Eq. (2.2) we obtain

$$u_1 = A(t)x_1, \quad u_2 = -A(t)x_2 \tag{2.5}$$

and from Eq. (1.9) we find

$$\frac{p}{\rho} = -\frac{c_1^2(t)}{2}(x_1^2 + x_2^2) - c_1'(t)\left(\frac{x_1^2}{2} - \frac{x_{10}^2}{2}\right) + c_1'(t)\left(\frac{x_2^2}{2} - \frac{x_{20}^2}{2}\right) + c_1(t)x_3 + c_2(t) \tag{2.6}$$

The general integral of system (1.10) in this case has the form

$$x_1x_2 = k_1$$

$$x_3 - \frac{1}{c_1(t)}\left(f(k_1) - \frac{1}{2}\int c_1(t)dt\right)\ln|x_1| + \frac{1}{4c_1(t)}\ln^2|x_1| = k_2$$

Relations (2.2) and (2.4)–(2.6) can be used to determine the distribution of the parameters of unsteady inviscid flow in the channel formed by hyperbolic cylinders

$$x_1x_2 = m_1, \quad x_1x_2 = m_2 \quad (m_1 \leqslant k_1 \leqslant m_2)$$

the generatrices of which are parallel to the x_3 axis (for given $c_1(t)$, $c_2(t)$, $f(x_1x_2)$).

3. THE SIMULATION OF STEADY FLOWS USING HARMONIC FUNCTIONS

Consider steady inviscid flow. The general solution of Eq. (1.8) in this case will be

$$u_3 = f(\psi(x_1, x_2)) - c_1\varphi(x_1, \psi(x_1, x_2)) \tag{3.1}$$

where

$$\varphi(x_1, k_1) = \int \frac{dx_1}{\partial\psi/\partial x_2} \quad \left(\frac{\partial\psi}{\partial x_2} = \frac{\partial\psi}{\partial x_2}(x_1, x_2(x_1, k_1))\right)$$

$\psi = \psi(x_1, x_2)$ is the solution of Eq. (1.4), $f(\psi)$ is an arbitrary continuously differentiable function, and the function $x_2 = x_2(x_1, k_1)$ is defined by the relation $\psi(x_1, x_2) = k_1$.

Relation (1.12) in this case has the form

$$x_3 + \frac{c_1}{2}\varphi^2(x_1, k_1) - f(k_1)\varphi(x_1, k_1) = k_2$$

The harmonic function ($\Delta\psi = 0$) will be the obvious solution of Eq. (1.4). Taking this into account solutions of well-studied problems on potential two-dimensional flows [1] can, in particular, be used to simulate the three-dimensional flows considered.

Consider, for example, the problem of the flow around a circular cylinder of radius a. As a solution of Eq. (1.4) we will take [1]

$$\psi = Ux_2\left(1 - \frac{a^2}{x_1^2 + x_2^2}\right), \quad U = \text{const} \tag{3.2}$$

Putting $c_1 = 0$ in Eq. (3.1), it can be assumed, in particular, that

$$u_3 = A\psi, \quad A = \text{const} \tag{3.3}$$

Using Eqs (3.2) and (3.3) we obtain the velocity distribution, and using Eq. (1.9), where we must put

$$\Delta\psi = 0, \quad \partial^2\psi/\partial t\partial x_1 = 0, \quad \partial^2\psi/\partial t\partial x_2 = 0$$

we obtain the pressure distribution.

The third-order curves

$$x_2\left(1 - \frac{a^2}{x_1^2 + x_2^2}\right) = k_1$$

are the directrices of the cylindrical stream surfaces in this case. On the surface of the cylinder $\psi = 0$.

Note that the harmonic function ψ is also the solution of Eq. (1.4) when the viscosity is taken into account; function (3.3) is the solution of Eq. (1.8) when $c_1 = 0$ in the steady case when the viscosity is taken into account. Thus, obviously, the harmonic function ψ and function (3.3) can be used to simulate three-dimensional viscous steady flows of a fluid in the regions bounded by the stream surfaces $\psi(x_1, x_2) = k_1$. Although the functions ψ and u_3 are harmonic, the flow will be turbulent.

4. THE SIMULATION OF STEADY FLOWS USING BIHARMONIC FUNCTIONS

If

$$\Delta\psi = c, \qquad c = \text{const} \tag{4.1}$$

then Eq. (1.4) is satisfied identically. It is analogous to the equation which describes the flow of a viscous fluid in a fixed cylindrical pipe.

Taking this into account, the well-known solutions of boundary-value problems for Eq. (4.1) [2–4] can be used to simulate the three-dimensional flows of an inviscid fluid in cylindrical pipes for various forms of cross-sections.

We will, for example, consider the steady inviscid flow of a fluid inside a fixed elliptic cylinder

$$x_1^2/a^2 + x_2^2/b^2 = 1$$

In this case the solution of Eq. (4.1), which vanishes at the boundary of the region, will be the function

$$\psi = M\left(1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2}\right), \quad M = -\frac{ca^2b^2}{2(a^2+b^2)} \tag{4.2}$$

Hence it follows that

$$u_1 = -2Mx_2/b^2, \quad u_2 = 2Mx_1/a^2$$

Confining ourselves to the case of positive values of x_2, found using Eq. (4.2) ($\psi(x_1, x_2) = k_1$), we obtain the function φ from Eq. (3.1) for u_3 in the given case

$$\varphi(x_1, k_1) = -\frac{ba}{2M}\arcsin\frac{x_1}{a\sqrt{1-k_1/M}}$$

We will obtain for the pressure from Eq. (1.9)

$$\frac{p}{\rho} + \frac{w^2}{2} = c\psi(x_1, x_2) + c_1x_3 + \text{const} \tag{4.3}$$

The biharmonic function $\psi(x_1, x_2)$, which satisfies Eq. (4.1), is also the solution of Eq. (1.4) taking into account the viscosity. Function (3.3) for $c_1/(\nu A) = c$ is the solution of Eq (1.8) taking into account the viscosity. Thus, obviously, the biharmonic function ψ and function (3.3) can be used to simulate three-dimensional viscous steady flows of fluid in regions, bounded by the stream surfaces $\psi(x_1, x_2) = k_1$.

Returning to steady inviscid flows, we will examine the flow in a cylindrical pipe of circular cross-section when the fluid is injected and sucked out through the walls. The boundary condition in this case has the form

$$\psi = f(\theta) \quad \text{for} \quad r = a, \quad 0 \leqslant \theta \leqslant 2\pi$$

where r and θ are polar coordinates ($x_1 = r\cos\theta, x_2 = r\sin\theta$) and a is the radius of the cylinder. Sections of the cylinder (for example, for $0 < \alpha_1 \leqslant \theta \leqslant \alpha_2 < 2\pi$) where $f(\theta) \equiv 0$, correspond to impenetrable walls.

We will seek a solution in the form

$$\psi = w + cr^2/4$$

where w is the solution of the homogeneous equation $\Delta w = 0$, satisfying the condition

$$w(a,\theta) = \varphi(\theta), \quad \varphi(\theta) = f(\theta) - ca^2/4 \quad \text{for} \quad r = a$$

Using the known solution for the function w [5] we will write the required solution

$$\psi(r,\theta) = \begin{cases} \dfrac{c}{4}r^2 + \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \varphi(\varsigma)\frac{a^2 - r^2}{r^2 - 2ar\cos(\theta - \varsigma) + a^2}\,d\varsigma, & r < a \\ \dfrac{c}{4}a^2 + \varphi(\theta), & r = a \end{cases} \tag{4.4}$$

For u_3 we can assume Eq. (3.3), in particular. We will have formula (4.3) for the pressure, where the function ψ is given by Eq. (4.4).

5. THE SIMULATION OF SYMMETRICAL FLOWS

Considering the steady flow of a viscous fluid we will consider cases where the flow possesses certain symmetries.

For example, suppose

$$\psi = \psi(r), \quad u_3 = u_3(r) \quad \left(r = \sqrt{x_1^2 + x_2^2}\right)$$

Then the left-hand sides of Eqs (1.4) and (1.8) vanish identically and instead of them we will have

$$\Delta\Delta\psi = 0, \qquad \Delta u_3 = c_1 \tag{5.1}$$

The function $\psi = \psi(r)$, which is the solution of the first equation of (5.1), is used in the plane case ($u_3 \equiv 0$) to describe the steady motion of the fluid between rotating coaxial cylinders [2], or in the channels formed by parts of coaxial circular cylinders [6].

The function which is the solution of the second equation of (5.1) has been used in [7] to describe the purely longitudinal motion of the fluid ($\psi \equiv 0$) in the annular channel between fixed cylinders, produced by a pressure difference at the channel ends.

The simultaneous use of the solutions of Eqs (5.1) enables us to consider the helical motion of a viscous fluid in the corresponding channels caused by the rotation of the cylinders and a pressure difference at the channel ends.

Suppose further that $\psi = \psi(\theta)$, $u_3 = u_3(\theta)$ (θ is the polar angle). In this case Eq. (1.4) takes the form

$$u'' + 4u + u^2/\nu - c = 0$$

where $u = \psi'$, $c = \text{const}$

Using the solution of this equation, obtained directly from the Navier–Stokes equation, the flow in a straight-wall diffuser [2] ($u_3 \equiv 0$) with the equations for the walls $\theta = \pm\alpha/2$ was considered by Hamel.

Putting $c_1 = 0$ in Eq. (1.8) we obtain

$$u_3 = A\theta + B, \qquad A, B = \text{const}$$

We obtain A and B by assuming the wall $\theta = -\alpha/2$ to be fixed and the wall $\theta = \alpha/2$ to be moving in the longitudinal direction with a constant velocity U:

$$u_3 = U(\theta/\alpha + \tfrac{1}{2}) \tag{5.2}$$

The simultaneous application of Hamel's relations and Eq. (5.2) enables us to consider the three-dimensional motion of the fluid in a channel with non-parallel plane walls.

Suppose now that $\psi = \psi(x_2)$, $u_3 = u_3(x_2)$. Considering that $u_1 = \partial\psi/\partial x_2$, using Eq. (1.4) we obtain a solution for u_1 in the form of a quadratic trinomial. From Eq. (1.8) we also obtain a solution for u_3 in the form of quadratic trinomial. Each of these relations by itself ($u_1 = u_1(x_2)$, $u_3 \equiv 0$; $u_1 \equiv 0$, $u_3 = u_3(x_2)$) describes plane Couette flow. The simultaneous use of the solutions $\psi = \psi(x_2)$, $u_3 = u_3(x_2)$ of Eqs (1.4) and (1.8) enables us to consider non-plane Couette flow between parallel moving planes. For this the boundary conditions for u_1 and u_2 on the walls must be chosen so that the vectors (a_1, a_3) and (b_1, b_3) are non-collinear, where a_1 and b_1 are the values of u_1 on the bottom and top planes, respectively, and a_3 and b_3 are the values of u_3 on the bottom and top planes, respectively.

6. THE SIMULATION OF VISCOUS FLOWS

Putting $c_1(t) \equiv 0$ and

$$u_3 = M\Delta\psi + N; \qquad M, N = \text{const} \tag{6.1}$$

in Eq. (1.8) we obtain an equation which is identical to (1.4). Hence, the solution of Eq. (1.4) simultaneously with the relation (6.1) enables us to simulate the three-dimensional viscous flow of a fluid.

Considering the steady viscous flow, we will take as the solution of Eq. (1.4) the function [2]

$$\psi = f(\varphi) + 2\nu a\chi$$

$$\varphi = -\frac{2(a\ln r + b\theta)}{a^2+b^2}, \quad \chi = \frac{2(b\ln r - a\theta)}{a^2+b^2}; \quad a, b = \text{const}$$

The following equation [2]

$$\varphi - \varphi_0 = \int [bu^3/(3\nu) + (a^2-b^2)u^2 + 2c_1u + c_2]^{-1/2} du$$

defines φ as a certain function of u. In this equation $u = f'(\varphi)$; φ_0, c_1, c_2, and c_1 and c_2 are constants.

In particular suppose $\varphi_0, c_1, c_2 = 0, a = b$. Then we obtain an explicit representation of the solutions of Eqs (1.4) and (1.8).

$$\psi = -\frac{12\nu}{b}\varphi^{-1} + 2\nu b\chi$$

$$\varphi = -\frac{\ln r + \theta}{b}, \quad \chi = \frac{\ln r - \theta}{b}, \quad \varphi < 0 \quad (r > 1, \ \theta > 0, \ b > 0)$$

$$u_3 = M\Delta\psi + N = \frac{48M\nu}{b^3}\frac{1}{r^2}\varphi^{-3} + N$$

Knowing ψ we obtain $u_\theta = -\partial\psi/\partial r$, $u_r = r^{-1}\partial\psi/\partial\theta$.

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Translated by V.S.